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Mutual Information of Ising Systems

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We obtain the mutual information of Ising systems, which shows singular behavior near the critical point. We connect the mutual information with the magnetization and the correlation function. The mutual information is a suitable measure for the critical behavior of Ising systems.

1. INTRODUCTION

Let AB be a joint system consisting of individual systems A and B. If A has states $\{\alpha\}$ and B has states $\{\beta\}$, AB has joint states $\{\alpha\beta\}$. The probability distributions of these systems are given by

$$p_\alpha^A = \sum_\beta p_{\alpha\beta}^{AB}, \qquad p_\beta^B = \sum_\alpha p_{\alpha\beta}^{AB} \tag{1}$$

where p_α^A and p_β^B denote the probabilities that A is in α and B is in β, respectively, and $p_{\alpha\beta}^{AB}$ denotes the joint probability that AB is in $\alpha\beta$. The mutual information (Shannon, 1948) between A and B is defined by

$$I_M(A : B) = S_A + S_B - S_{AB} \tag{2}$$

where S_A (S_B) is the individual entropy of A (B) and S_{AB} is the joint entropy of AB as follows:

$$S_A = -\sum_\alpha p_\alpha^A \log p_\alpha^A, \qquad S_B = -\sum_\beta p_\beta^B \log p_\beta^B \tag{3}$$

$$S_{AB} = -\sum_{\alpha\beta} p_{\alpha\beta}^{AB} \log p_{\alpha\beta}^{AB} \tag{4}$$

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Equation (2) can be applied, e.g., to complex systems and used in measuring complexity (Grassberger, 1986; Wackerbauer *et al.*, 1994).

The mutual information means the common information of two systems which has the property

$$0 \le I_M(A:B) \le \begin{cases} S_A & \text{if } S_A < S_B \\ S_B & \text{if } S_B < S_A \end{cases} \tag{5}$$

The condition of independent joint probability

$$p^{AB}_{\alpha\beta} = p^A_\alpha p^B_\beta \tag{6}$$

leads to $I_M(A:B) = 0$. If one system is completely dependent on the other system, $I_M(A:B)$ takes its maximum. Thus the mutual information can be considered as a measure of dependence like the correlation function. As for the difference between the mutual information and the correlation function (Fraser and Swinney, 1986; Fraser, 1989; Li, 1990), it should be noted that while the correlation function is only the measure of the linear dependence, the mutual information provides the general dependence between two systems.

In this paper we apply the mutual information to Ising systems in order to see the effect of a phase transition, by considering joint states of two spin systems. The behavior of the mutual information can be estimated from the relation to the magnetization and the correlation function. It turns out that the value of the mutual information is strongly dependent on the values of the individual entropies for the variation of the temperature, as given by the inequality (5). This constraint yields an interesting feature of the mutual information in terms of the temperature which characterizes the phase transition of spin systems.

In the next section, we give formulas for the mutual information for Ising systems. In Section 3, we show that the mutual information has a characteristic feature near the critical temperature. Section 4 is devoted to our conclusions.

2. MUTUAL INFORMATION OF ISING SYSTEMS

Let us consider a d-dimensional square lattice with L^d sites. The Hamiltonian is given by

$$H = -\sum_{\langle ij \rangle} s_i s_j \tag{7}$$

where s_i is a spin variable which takes the values ± 1, and $\langle ij \rangle$ goes over all the nearest neighbor pairs on the lattice. The system depends on the temperature T through the Boltzmann weight factor $\exp(-H/kT)$, where k is the Boltzmann constant. Here we use units such that of $k = 1$, for the sake of simplicity.

To this system we apply the mutual information in the following manner. Let p_s ($s = \uparrow, \downarrow$) be the probability for the spin state s on the whole lattice,

$$p_\uparrow = \frac{N_\uparrow}{N_\uparrow + N_\downarrow}, \qquad p_\downarrow = \frac{N_\downarrow}{N_\uparrow + N_\downarrow} \tag{8}$$

where $N_\uparrow$ ($N_\downarrow$) is the number of up (down) spins. Moreover, we assign the probability $p_{ss'}(l)$ ($s, s' = \uparrow, \downarrow$) to the joint state ss' of two spins specified by the distance l as follows:

$$p_{\uparrow\uparrow}(l) = \frac{N_{\uparrow\uparrow}(l)}{N}, \qquad p_{\downarrow\downarrow}(l) = \frac{N_{\downarrow\downarrow}(l)}{N}, \qquad p_{\uparrow\downarrow}(l) = p_{\downarrow\uparrow}(l) = \frac{N_{\uparrow\downarrow}(l)}{N} \tag{9}$$

with

$$N = N_{\uparrow\uparrow}(l) + N_{\downarrow\downarrow}(l) + 2N_{\uparrow\downarrow}(l)$$

where $N_{\uparrow\uparrow}(l)$, $N_{\downarrow\downarrow}(l)$, and $N_{\uparrow\downarrow}(l)$ are the numbers of the up–up pairs, down–down pairs, and up–down pairs, respectively. These probabilities are connected through the relation

$$p_s = \sum_{s'} p_{ss'}(l) = \sum_{s'} p_{s's}(l) \tag{10}$$

The mutual information between the two spins can be defined by

$$I_M(l) = -2 \sum_s p_s \log p_s + \sum_{s,s'} p_{ss'}(l) \log p_{ss'}(l) \tag{11}$$

We can easily see that this quantity (11) falls to zero with increase of l, because the joint probability becomes independent following condition (6) for larger l. We also get $I_M(l) = 0$ in both limits $T \to \infty$ and $T \to 0$.

Now let us obtain the relation of the mutual information (11) to the magnetization M and the correlation function $\Gamma(l)$. Since they are defined as

$$M = \sum_s s p_s \tag{12}$$

and

$$\Gamma(l) = \sum_{s,s'} ss' p_{ss'}(l) \tag{13}$$

by using the normalization condition $\Sigma_s\, p_s = \Sigma_{ss'}\, p_{ss'} = 1$, we have the following expressions:

$$p_\uparrow = \frac{1 + M}{2}, \qquad p_\downarrow = \frac{1 - M}{2} \tag{14}$$

and

$$p_{\uparrow\uparrow}(l) = \frac{1 + 2M + \Gamma(l)}{4}$$

$$p_{\downarrow\downarrow}(l) = \frac{1 - 2M + \Gamma(l)}{4}$$

$$p_{\uparrow\downarrow}(l) = p_{\downarrow\uparrow}(l) = \frac{1 - \Gamma(l)}{4} \tag{15}$$

Inserting (14) and (15) into equation (11), we have

$$\begin{aligned} I_M(l) = &\frac{1}{4}\log\frac{(1-\Gamma(l))^2(1-2M+\Gamma(l))(1+2M+\Gamma(l))}{(1-M^2)^4} \\ &+ \frac{M}{2}\log\frac{(1-M)^2(1+2M+\Gamma(l))}{(1+M)^2(1-2M+\Gamma(l))} \\ &+ \frac{\Gamma(l)}{4}\log\frac{(1+2M+\Gamma(l))(1-2M+\Gamma(l))}{(1-\Gamma(l))^2} \end{aligned} \tag{16}$$

This relation yields the characteristic behavior of $I_M(l)$. For example, it follows that when T is above the critical temperature, where $M = 0$ and $\Gamma(l)$ is small, $I_M(l)$ behaves like

$$I_M(l) \simeq \frac{1}{2}\Gamma(l)^2 \sim \frac{1}{2}\exp\left(-\frac{2l}{\xi}\right) \tag{17}$$

where ξ is the correlation length.

3. SINGULARITY OF THE MUTUAL INFORMATION

In order to see the behavior of the mutual information around the critical temperature T_c, we perform the differentiation of equation (16) with respect to T. In the region of just below T_c we have power-low behavior of the magnetization, i.e., $M \sim (T_c - T)^\beta$, where β is the critical index. The results are given for $T > T_c$ and $T < T_c$ as follows:

$$\frac{\partial I_M(l)}{\partial T} = \frac{1}{2}\frac{\partial\Gamma(l)}{\partial T}\log\frac{1+\Gamma(l)}{1-\Gamma(l)} < 0 \qquad (T > T_c) \tag{18}$$

and

$$\left.\frac{\partial I_M(l)}{\partial T}\right|_{T\to T_c} \sim \beta(T_c - T)^{2\beta-1}|_{T\to T_c} \to +\infty \qquad (T < T_c,\ \beta < 1/2) \tag{19}$$

Since $0 < \Gamma < 1$ and $\partial\Gamma/\partial T < 0$, equation (18) is negative. Equation (19) is valid if $\beta < 1/2$. The theoretical study of critical indices gives $\beta = 1/8$ for $d = 2$ and $\beta \simeq 0.31$ for $d = 3$ (Wilson and Kogut, 1974). Therefore, we see in both cases that the mutual information has a sharp peak at the critical temperature, which shows the singular behavior of the spin systems.

Figure 1 shows the temperature dependence of $I_M(l)$ calculated for 64 $\times$ 64 lattice, where $l = 1$. The peak position is slightly shifted from the critical temperature ($T_c^{-1} = 0.441$) because of the finite-size effect. In addition,

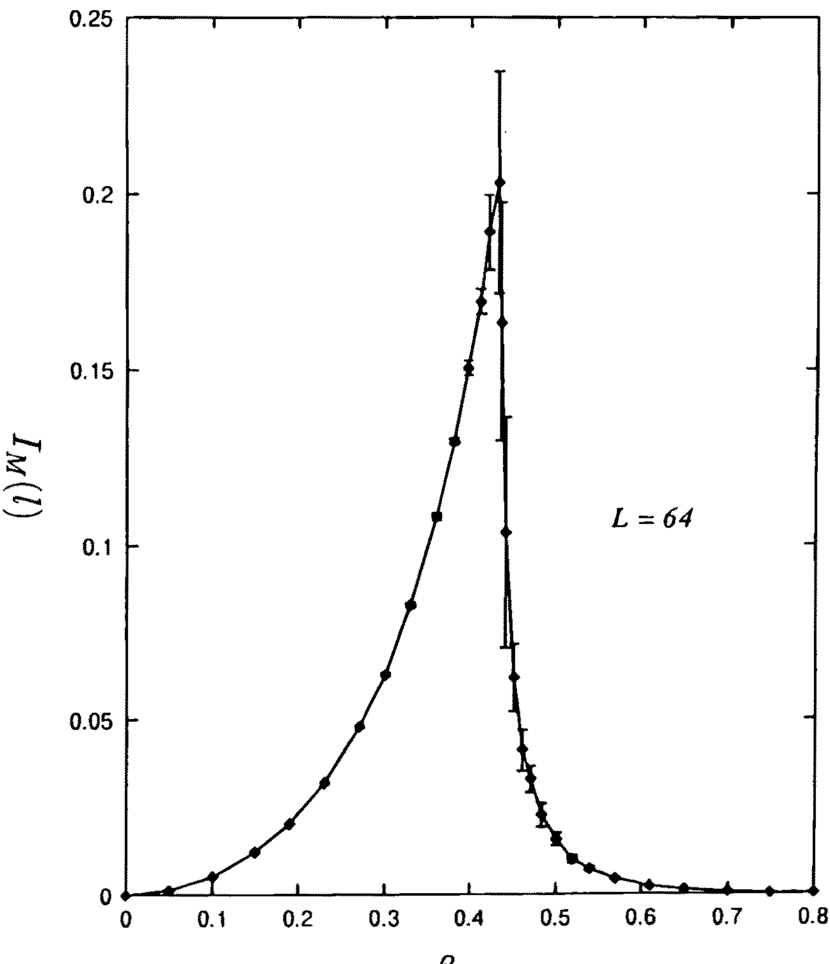

Fig. 1. Dependence of the mutual information $\langle I_M(l)\rangle$ on the temperature T for a 64×64 lattice, where $l = 1$.

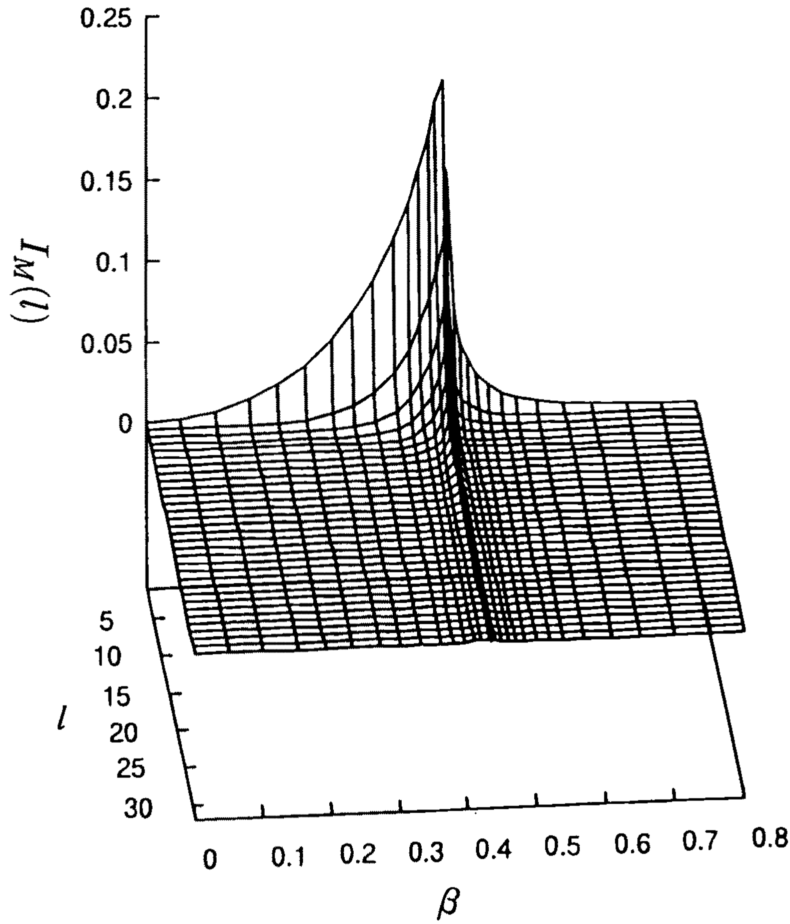

Fig. 2. Dependence of the mutual information $\langle I_M(l)\rangle$ on the temperature T and the distance l for a 64×64 lattice.

we show in Fig. 2 the dependence of the mutual information on the temperature T and the distance l.

4. CONCLUSION

In this paper we have applied the mutual information to Ising systems by considering joint states of two spin systems, and shown their characteristic behaviors as a function of the temperature. It turns out from the relation to the magnetization and the correlation function that the mutual information has a sharp peak at the critical temperature. Thus, we conclude that the mutual information can be used as the suitable measure for the phase transition of spin systems, and furthermore for various systems in which probability distributions are defined definitely.

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